Example 26 A line makes angles α, β, γ and δ with the diagonals of a cube, prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

Solution A cube is a rectangular parallelopiped having equal length, breadth and height. Let OADBFEGC be the cube with each side of length a units. (Fig 11.21) The four diagonals are OE, AF, BG and CD.

The direction cosines of the diagonal OE which is the line joining two points O and E are

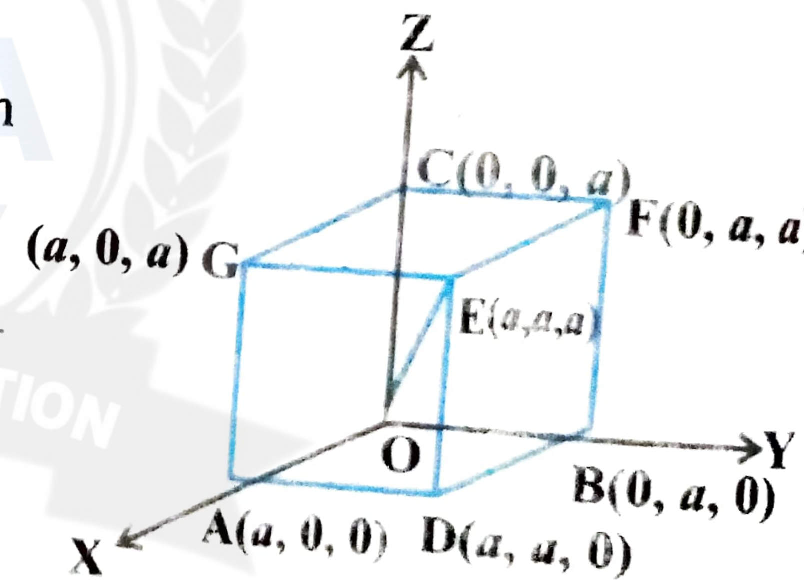

Fig 11.21

$$\frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}$$

i.e., $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Similarly, the direction cosines of AF, BG and CD are $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$; $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$, respectively.

Let l, m, n be the direction cosines of the given line which makes angles $\alpha, \beta, \gamma, \delta$ with OE, AF, BG, CD, respectively. Then

$$\cos\alpha = \frac{1}{\sqrt{3}}(l + m + n);\ \cos\beta = \frac{1}{\sqrt{3}}(-l + m + n);$$

$$\cos\gamma = \frac{1}{\sqrt{3}}(l - m + n);\ \cos\delta = \frac{1}{\sqrt{3}}(l + m - n) \quad \text{(Why?)}$$

Squaring and adding, we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

$$= \frac{1}{3}\left[(l + m + n)^2 + (-l + m + n)^2] + (l - m + n)^2 + (l + m - n)^2 \right]$$

$$= \frac{1}{3}[4(l^2 + m^2 + n^2)] = \frac{4}{3} \quad (\text{as } l^2 + m^2 + n^2 = 1)$$

Example 27 Find the equation of the plane that contains the point (1, – 1, 2) and is perpendicular to each of the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$.

Solution The equation of the plane containing the given point is

$$A(x - 1) + B(y + 1) + C(z - 2) = 0 \quad \dots (1)$$

Applying the condition of perpendicularly to the plane given in (1) with the planes

$2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$, we have

$$2A + 3B - 2C = 0 \text{ and } A + 2B - 3C = 0$$

Solving these equations, we find A = – 5C and B = 4C. Hence, the required equation is

$$-5C(x - 1) + 4C(y + 1) + C(z - 2) = 0$$

i.e. $$5x - 4y - z = 7$$

Example 28 Find the distance between the point P(6, 5, 9) and the plane determined by the points A (3, – 1, 2), B (5, 2, 4) and C(– 1, – 1, 6).

Solution Let A, B, C be the three points in the plane. D is the foot of the perpendicular drawn from a point P to the plane. PD is the required distance to be determined, which is the projection of $\overrightarrow{AP}$ on $\overrightarrow{AB} \times \overrightarrow{AC}$.

... the dot product of $\overrightarrow{AP}$ with the unit vector along $\overrightarrow{AB} \times \overrightarrow{AC}$

So
$$\overrightarrow{AP} = 3\hat{i} + 6\hat{j} + 7\hat{k}$$

and
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 12\hat{i} - 16\hat{j} + 12\hat{k}$$

Unit vector along $\overrightarrow{AB} \times \overrightarrow{AC} = \dfrac{3\hat{i} - 4\hat{j} + 3\hat{k}}{\sqrt{34}}$

Hence
$$PD = (3\hat{i} + 6\hat{j} + 7\hat{k}) \cdot \frac{3\hat{i} - 4\hat{j} + 3\hat{k}}{\sqrt{34}}$$

$$= \frac{3\sqrt{34}}{17}$$

Alternatively, find the equation of the plane passing through A, B and C and compute the distance of the point P from the plane.

Example 29 Show that the lines

$$\frac{x - a + d}{\alpha - \delta} = \frac{y - a}{\alpha} = \frac{z - a - d}{\alpha + \delta}$$

and
$$\frac{x - b + c}{\beta - \gamma} = \frac{y - b}{\beta} = \frac{z - b - c}{\beta + \gamma}$$ are coplanar.

Solution

Here

$x_1 = a - d$	$x_2 = b - c$
$y_1 = a$	$y_2 = b$
$z_1 = a + d$	$z_2 = b + c$
$a_1 = \alpha - \delta$	$a_2 = \beta - \gamma$
$b_1 = \alpha$	$b_2 = \beta$
$c_1 = \alpha + \delta$	$c_2 = \beta + \gamma$

Now consider the determinant

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} b - c - a + d & b - a & b + c - a - d \\ \alpha - \delta & \alpha & \alpha + \delta \\ \beta - \gamma & \beta & \beta + \gamma \end{vmatrix}$$

Adding third column to the first column, we get

$$2\begin{vmatrix} b-a & b-a & b+c-a-d \\ \alpha & \alpha & \alpha+\delta \\ \beta & \beta & \beta+\gamma \end{vmatrix} = 0$$

Since the first and second columns are identical. Hence, the given two lines are coplanar.

Example 30 Find the coordinates of the point where the line through the points A (3, 4, 1) and B (5, 1, 6) crosses the XY-plane.

Solution The vector equation of the line through the points A and B is

$$\vec{r} = 3\hat{i} + 4\hat{j} + \hat{k} + \lambda[(5-3)\hat{i} + (1-4)\hat{j} + (6-1)\hat{k}]$$

i.e. $$\vec{r} = 3\hat{i} + 4\hat{j} + \hat{k} + \lambda(2\hat{i} - 3\hat{j} + 5\hat{k}) \quad \ldots (1)$$

Let P be the point where the line AB crosses the XY-plane. Then the position vector of the point P is of the form $x\hat{i} + y\hat{j}$.

This point must satisfy the equation (1). (Why ?)

i.e. $$x\hat{i} + y\hat{j} = (3+2\lambda)\hat{i} + (4-3\lambda)\hat{j} + (1+5\lambda)\hat{k}$$

Equating the like coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we have

$$x = 3 + 2\lambda$$
$$y = 4 - 3\lambda$$
$$0 = 1 + 5\lambda$$

Solving the above equations, we get

$$x = \frac{13}{5} \text{ and } y = \frac{23}{5}$$

Hence, the coordinates of the required point are $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$.

Summary

- **Direction cosines of a line** are the cosines of the angles made by the line with the positive directions of the coordinate axes
- If l, m, n are the direction cosines of a line, then $l^2 + m^2 + n^2 = 1$.
- Direction cosines of a line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

$$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$$

where $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

- **Direction ratios of a line** are the numbers which are proportional to the direction cosines of a line.
- If l, m, n are the direction cosines and a, b, c are the direction ratios of a line

then

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}} ; m = \frac{b}{\sqrt{a^2+b^2+c^2}} ; n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

- **Skew lines** are lines in space which are neither parallel nor intersecting. They lie in different planes.
- **Angle between skew lines** is the angle between two intersecting lines drawn from any point (preferably through the origin) parallel to each of the skew lines.
- If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two lines; and θ is the acute angle between the two lines; then

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$$

- If a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of two lines and θ is the acute angle between the two lines; then

$$\cos\theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

- Vector equation of a line that passes through the given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda \vec{b}$.
- Equation of a line through a point (x_1, y_1, z_1) and having direction cosines l, m, n is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

- The vector equation of a line which passes through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda (\vec{b} - \vec{a})$.
- Cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$.
- If θ is the acute angle between $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$, then

$$\cos\theta = \left| \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1|\,|\vec{b}_2|} \right|$$

- If $\dfrac{x - x_1}{l_1} = \dfrac{y - y_1}{m_1} = \dfrac{z - z_1}{n_1}$ and $\dfrac{x - x_2}{l_2} = \dfrac{y - y_2}{m_2} = \dfrac{z - z_2}{n_2}$

are the equations of two lines, then the acute angle between the two lines is given by $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$.

Shortest distance between two skew lines is the line segment perpendicular to both the lines.

Shortest distance between $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ is

$$\left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

Shortest distance between the lines: $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is

$$\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

Distance between parallel lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}$ is

$$\left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right|$$

In the vector form, equation of a plane which is at a distance d from the origin, and $\hat{n}$ is the unit vector normal to the plane through the origin is $\vec{r} \cdot \hat{n} = d$.

Equation of a plane which is at a distance of d from the origin and the direction cosines of the normal to the plane as l, m, n is $lx + my + nz = d$.

The equation of a plane through a point whose position vector is $\vec{a}$ and perpendicular to the vector $\vec{N}$ is $(\vec{r} - \vec{a}) . \vec{N} = 0$.

Equation of a plane perpendicular to a given line with direction ratios A, B, C and passing through a given point (x_1, y_1, z_1) is

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

Equation of a plane passing through three non collinear points (x_1, y_1, z_1),

(x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

- Vector equation of a plane that contains three non collinear points having position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$
- Equation of a plane that cuts the coordinates axes at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

- Vector equation of a plane that passes through the intersection of planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is $\vec{r} \cdot (\vec{n}_1 + \lambda \vec{n}_2) = d_1 + \lambda d_2$, where λ is any nonzero constant.
- Cartesian equation of a plane that passes through the intersection of two given planes $A_1 x + B_1 y + C_1 z + D_1 = 0$ and $A_2 x + B_2 y + C_2 z + D_2 = 0$ is $(A_1 x + B_1 y + C_1 z + D_1) + \lambda(A_2 x + B_2 y + C_2 z + D_2) = 0$.
- Two lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ are coplanar if

$$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 0$$

- In the cartesian form above lines passing through the points A (x_1, y_1, z_1) and B (x_2, y_2, z_2)

$$= \frac{y - y_2}{b_2} = \frac{z - z_2}{C_2} \text{ are coplanar if } \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

- In the vector form, if θ is the angle between the two planes, $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$, then $\theta = \cos^{-1} \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$
- The angle ϕ between the line $\vec{r} = \vec{a} + \lambda \vec{b}$ and the plane $\vec{r} \cdot \hat{n} = d$ is

$$\sin \phi = \left| \frac{\vec{b} \cdot \hat{n}}{|\vec{b}||\hat{n}|} \right|$$

- The angle θ between the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is given by

$$\cos \theta = \left| \frac{A_1 A_2 + B_1 B_2 + C_1 C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\ \sqrt{A_2^2 + B_2^2 + C_2^2}} \right|$$

- The distance of a point whose position vector is $\vec{a}$ from the plane $\vec{r} \cdot \hat{n} = d$ is $|d - \vec{a} \cdot \hat{n}|$
- The distance from a point (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is

$$\left| \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}} \right|.$$